Some extremal problems on the plane

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Outline

One problem by Tiberiu Popoviciu

The first circle of questions is connected with one problem, posed by the eminent Romanian mathematician, Tiberiu Popoviciu (1906–1975), see [22].[1] Let $ABCD$ be any convex quadrilateral in the Euclidean plane. Let us consider the points A_1, B_1, C_1, and D_1 of the segments AB, BC, CD, and DA respectively, such that

$$\frac{|AA_1|}{|A_1B|} = \frac{|BB_1|}{|B_1C|} = \frac{|CC_1|}{|C_1D|} = \frac{|DD_1|}{|D_1A|} = 1.$$

The straight lines AB_1, BC_1, CD_1, DA_1 form a quadrilateral $KLMN$ situated inside $ABCD$. Let us denote by S and s the areas of the quadrilaterals $ABCD$ and $KLMN$ respectively. The problem consists in proving the inequality

$$\frac{1}{6}S \leq s \leq \frac{1}{5}S$$

and in the study of cases of equalities, see Fig. 1 a). Naturally, the question arises as to where the quantities $1/6$ and $1/5$ come from in the formulation of the above problem? They appeared thanks to special examples.

[1] T. Popoviciu, *Problem 5897*, Gazeta Matematică, 49 (1943), P. 322.

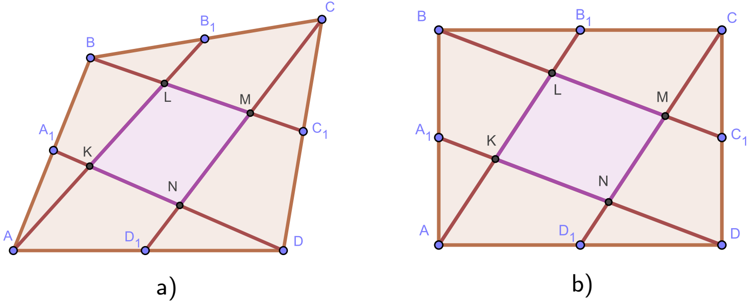

Fig. 1: a) The original Popoviciu's problem; b) An example for the Popoviciu's problem with $S = 5s$.

We have $S = 5s$ in the case when $ABSD$ is a parallelogram (a rectangle or a square in particular), see Fig. 1 b). The equality $S = 6s$ we obtain if two adjacent vertices of $ABCD$ are coincide. In particular, if $C = B$, then $B_1 = L = M = C = B$ and $K = A_1$, see Fig. 2 a).

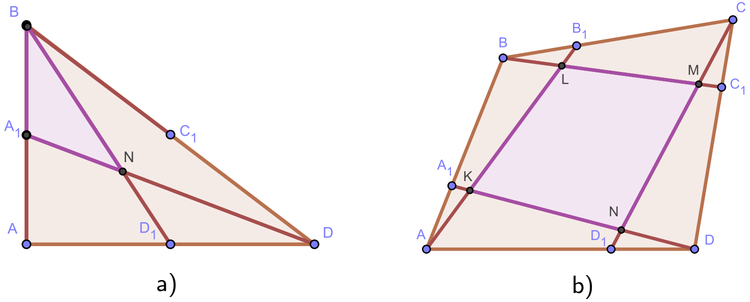

Fig. 2: a) An example for the Popoviciu's problem with $S = 6s$; b) The generalized Popoviciu's problem.

The solution of this problem was obtained by Yu.G. Nikonorov in [17],[2] whereas Yu.G. Nikonorov and Yu.V. Nikonorova solved some natural generalization of the Popoviciu's problem in [19].[3] Later the same results were obtained by R. Mabry in [13][4] (the original Popoviciu's problem) and by J.M. Ash, M.A. Ash, and P.F. Ash in [1][5] (the generalized Popoviciu's problem).

We will consider the solution of the generalized Popoviciu's problem and discuss tools that could be applied in the studying of similar problems.

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[2] Yu.G. Nikonorov, *Some problems of Euclidean geometry,* Preprint, Rubtsovsk Industrial Institute, Rubtsovsk, 1998, 32 p.

[3] Yu.G. Nikonorov, Yu.V. Nikonorova, *Generalized Popoviciu's problem*, Tr. Rubtsovsk. Ind. Inst., 7 (2000), 229–232 (in Russian) (2000), **Zbl.** 0958.51021. English translation: arXiv:1806.03345.

[4] R. Mabry. *Crosscut convex quadrilaterals*, Math. Mag., 84(1) (2011), 16–25, **Zbl.**1227.51015, **MR**2793173.

[5] J.M. Ash, M.A. Ash, P.F. Ash, *Constructing a quadrilateral inside another one,* The Mathematical Gazette, 93(528), (2009) 522–532, see also arXiv:0704.2716.

Now, let us consider *the generalized Popoviciu's problem.* Let $ABCD$ be any convex quadrilateral in the Euclidean plane. Let us consider the points A_1, B_1, C_1, and D_1 of the segments AB, BC, CD, and DA respectively, such that

$$\frac{|AA_1|}{|A_1B|} = \frac{|BB_1|}{|B_1C|} = \frac{|CC_1|}{|C_1D|} = \frac{|DD_1|}{|D_1A|} = k$$

for some fixed $k > 0$. The straight lines AB_1, BC_1, CD_1, DA_1 form a quadrilateral $KLMN$ (K, L, M, N are the intersection points for the first and the fourth, the first and the second, the second and the third, the third and the fourth straight lines respectively), situated inside $ABCD$, see Fig. 2 b).
Let us denote by S and s the areas of the quadrilaterals $ABCD$ and $KLMN$ respectively. The generalized Popoviciu's problem consists in finding exact upper and lower bounds for the ratio s/S for any given $k > 0$.

Theorem ([19])

For an arbitrary convex quadrilateral in the Euclidean plane and for any $k > 0$, the inequality

$$\frac{1}{(k+1)(k^2+k+1)} \cdot S \le s \le \frac{1}{2k^2+2k+1} \cdot S$$

holds. Moreover, if $S > 0$ then the equality $(k+1)(k^2+k+1)\, s = S$ is fulfilled exactly for quadrilaterals with two coinciding vertices, whereas the set of quadrilaterals with the property $(2k^2+2k+1)\, s = S$ contains all parallelograms, but not only them.

We may assume that the vertices A, B, and D are pairwise distinct points. Using a suitable affine transformation (the ratio of the areas does not change in this case), one can reduce the problem to the case, when $\angle BAD$ of the quadrilateral $ABCD$ is right, and the sides AB and AD have unit length. We introduce a Cartesian coordinate system in the plane, taking the point A as the origin and the rays AD and AB as the coordinate rays. In this coordinate system, the points A, B, D, C have coordinates $(0,0)$, $(0,1)$, $(1,0)$, (a,b) respectively, where $a \geq 0$, $b \geq 0$, $a+b \geq 1$. Let us consider the set

$$\Omega := \{(a,b) \in \mathbb{R}^2 \mid a \geq 0, \, b \geq 0, \, a+b \geq 1\}.$$

The values s and S can be calculated using standard analytical geometry tools. It is easy to get that $2S = a + b$. It is a simple problem to calculate the coordinates of the following points:

$$A_1 = \left(0, \frac{k}{k+1}\right), \qquad B_1 = \left(\frac{ak}{k+1}, \frac{1+kb}{k+1}\right),$$
$$C_1 = \left(\frac{a+k}{k+1}, \frac{b}{k+1}\right), \qquad D_1 = \left(\frac{1}{k+1}, 0\right).$$

It is easy also to find the equations of the straight lines DA_1, CD_1, AB_1, and BC_1:

$$\begin{aligned} kx + (k+1)y - k &= 0, \\ (k+1)bx + (1 - a(k+1))y - b &= 0, \\ (kb+1)x - kay &= 0, \\ (k+1-b)x + (a+k)y - (a+k) &= 0. \end{aligned}$$

Now, we should calculate the coordinates of the points K, L, M, and N, which are intersection points of pairs of the corresponding straight lines. Omitting the standard calculations, we obtain

$$K = \left(\frac{ak^2}{ak^2+bk^2+bk+k+1}, \frac{k(bk+1)}{ak^2+bk^2+bk+k+1}\right),$$

$$L = \left(\frac{ak(a+k)}{ak^2+bk^2+ak+k+a}, \frac{(bk+1)(a+k)}{ak^2+bk^2+ak+k+a}\right),$$

$$M = \left(\frac{ak^2+a^2k+ak+bk-k+a^2+ab-a}{ak^2+bk^2+2ak+bk-k+a+b-1}, \frac{b\left(k^2+ak+a+b-1\right)}{ak^2+bk^2+2ak+bk-k+a+b-1}\right),$$

$$N = \left(\frac{ak^2+ak+bk-k+b}{ak^2+bk^2+ak+2bk-k+b}, \frac{bk^2}{ak^2+bk^2+ak+2bk-k+b}\right).$$

It is obvious that

$$2s = 2S_{\Delta ANM} + 2S_{\Delta AML} - 2S_{\Delta ANK}.$$

Using determinants for calculating the areas of the triangles, we obtain

$$2S_{\Delta AML} = (a+k)\cdot\frac{ak^2-bk^2-k+ak^2b+ak+a^2k+b^2k^2+bk-a+ab+a^2}{(-k+ak^2+2ak-1+a+b+bk^2+bk)(ak^2+ak+a+bk^2+k)},$$

$$2S_{\Delta ANM} = b\cdot\frac{ak^2b+k-b-2bk+ab-2ak-ak^2+bk^2+b^2k+a^2k+b^2+3bka+a^2k^2}{(bk^2+2bk+b-k+ak^2+ak)(-k+ak^2+2ak-1+a+b+bk^2+bk)},$$

$$2S_{\Delta ANK} = k\cdot\frac{b^2k^2+b^2k-bk^2+ak^2b+bk+b-k+ak^2+ak}{(bk^2+bk+k+1+ak^2)(bk^2+2bk+b-k+ak^2+ak)}.$$

Therefore,

$$\frac{s}{S}=\frac{P(a,b)}{Q(a,b)},$$

where

$$
\begin{aligned}
P(a,b) = {} & -2a^2k^2b + 6ab^4k^3 - 4ak^2b + 12a^2k^3b^2 + 9ab^4k^5 \\
& +16a^2k^4b^2 + 19a^2k^4b^3 + 9a^2b^3k^2 + 8a^3k^2b^2 + 17a^3k^3b^2 + 2a^2k^6b \\
& +2a^4k^6b + 8a^3k^5b - b^2a - 3b^2k^2 + a^2k + bk^2 + ak^2 - 8ab^3k^5 + 9a^4k^4b \\
& +2b^3ak - 4a^2k^5b + 6a^3kb + 3b^4k^3 - 5a^2kb + 3ab^4k^2 + 14a^3k^5b^2 \\
& +18a^2k^5b^3 - 4a^2k^5b^2 + 6a^3k^6b^2 + 15a^2k^3b^3 + 11ab^4k^4 + ab^2k^2 - 9ak^3b^2 \\
& +12a^3k^4b + 4ab^3k^4 + 8a^3k^2b + a^4k^3 + 3a^4k^2 - 4a^2k^6b^2 - 3a^3k^2 + 8a^4k^3b \\
& +a^3k^3 - 2a^2k^3 - 5a^3k^4 + 6a^2k^6b^3 + 2b^2k^3 - a^2b - 2a^3k + 2ab^4k^6 \\
& +17a^3k^4b^2 + 5a^4k^5b + a^4k^4 + a^5k^5 + 4a^3k^3b + 12ab^3k^3 + 4ab^3k^2 \\
& +7a^2k^2b^2 - 2b^2ka + 7a^2b^2k + 8ak^4b + 8ak^5b^2 - 5b^3k^3 + b^3k^4 + 4b^3k^5 \\
& -2bk^4 + 2b^2k^4 + 6a^2k^4 - 4b^2k^5 - 2ak^4 + 4a^2k^5 + a^4k + 2a^2b^2 + a^3b \\
& +2ak^6b^2 - 11ak^4b^2 - 3a^2k^3b - 17a^2k^4b - b^2k + 2a^4k^6 + 4a^4k^5 - 8a^3k^5 \\
& +2a^5k^4 + 2b^4k^6 - b^4k^4 - a^2k^2 + a^5k^3 + a^3b^2k - 2a^3k^6 + 2a^2b^3k \\
& -2b^3k^6 + 2a^4k^2b + b^4ak + b^3a + b^5k^3 + 2b^4k^2 + b^3k + 2b^5k^4 + b^5k^5,
\end{aligned}
$$

$$\begin{aligned} Q(a,b) &= (a+b)\left(ak^2+ak+a+bk^2+k\right) \\ &\times \left(-k+ak^2+2ak-1+a+b+bk^2+bk\right) \\ \times \left(bk^2+bk+k+1+ak^2\right) &\left(bk^2+2bk+b-k+ak^2+ak\right). \end{aligned}$$

It should be noted that

$$\begin{aligned} &-k+ak^2+2ak-1+a+b+bk^2+bk \\ &= (a+b-1)+k(a+b-1)+ak+k^2(a+b)>0, \\ bk^2+2bk+b-k+ak^2+ak &= k(a+b-1)+bk+b+k^2(a+b)>0 \end{aligned}$$

on the set Ω. This implies that $Q(a,b)>0$ on the set Ω.

Therefore, the proof of the theorem reduces to the proof of the following two inequalities for $(a,b)\in\Omega$:

1) $(k+1)(k^2+k+1)P(a,b)-Q(a,b)\geq 0$,

2) $Q(a,b)-(2k^2+2k+1)P(a,b)\geq 0$,

and the study of the equality cases.

$$\begin{aligned} Q(a,b) = (a+b)\left(ak^2+ak+a+bk^2+k\right) \\ \times\left(-k+ak^2+2ak-1+a+b+bk^2+bk\right) \\ \times\left(bk^2+bk+k+1+ak^2\right)\left(bk^2+2bk+b-k+ak^2+ak\right). \end{aligned}$$

It should be noted that

$$\begin{aligned} -k+ak^2+2ak-1+a+b+bk^2+bk \\ =(a+b-1)+k(a+b-1)+ak+k^2(a+b)>0, \\ bk^2+2bk+b-k+ak^2+ak = k(a+b-1)+bk+b+k^2(a+b)>0 \end{aligned}$$

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Let us consider the first inequality. Direct calculations show that

$$\begin{aligned}(k+1)(k^2+k+1)P(a,b)-Q(a,b) &= k^3\,(a+b)\left(1+2k+2k^2\right)\\ &\times\left(ak^2b+ak^2+2ak+a+b-1-k+b^2k-bk^2+b^2k^2\right)\\ &\times\left(a^2k^2+a^2k+ba-2ak+k+2bak-ak^2+ak^2b+bk^2\right).\end{aligned}$$

Note that

$$\begin{aligned}&ak^2b+ak^2+2ak+a+b-1-k+b^2k-bk^2+b^2k^2\\ &\qquad=(a+b-1)(k^2b+1)+ak^2+(2a+b^2-1)k,\\ &a^2k^2+a^2k+ba-2ak+k+2bak-ak^2+ak^2b+bk^2\\ &\qquad=ak^2(a+b-1)+ab+bk^2+(a^2+2ab-2a+1)k.\end{aligned}$$

We note that the inequality $a+b\geq 1$ is fulfilled on the set Ω. Thus, to prove the first part of the theorem it suffices to make sure that the inequalities $2a+b^2-1\geq 0$ and $a^2+2ab-2a+1\geq 0$ are fulfilled on the set Ω.

Let us prove the first inequality. We know that $b^2+1 \geq 2b$ ($b^2+1=2b$ if and only if $b=1$), hence,

$$2a+b^2-1 \geq 2a+(2b-1)-1 = 2(a+b-1) \geq 0.$$

It is easy to see that $2a+b^2-1=0$ for $(a,b)\in\Omega$ if and only if $(a,b)=(0,1)$. Note also that

$$(a+b-1)(k^2b+1)+ak^2+(2a+b^2-1)k=0$$

for $(a,b)=(0,1)$.

Let us prove the second inequality. Obviously, the inequality $a^2+2ab-2a+1=(a-1)^2+2ab \geq 0$ is fulfilled on the set Ω, and the equality is achieved only when $(a,b)=(1,0)$. Note that

$$ak^2(a+b-1)+ab+bk^2+(a^2+2ab-2a+1)k=0$$

for $(a,b)=(1,0)$.

Finally, we get that

$$(k+1)(k^2+k+1)P(a,b) - Q(a,b) \geq 0$$

on the set Ω, and the equality is achieved only when $(a,b) = (1,0)$ or $(a,b) = (0,1)$, that is, in the case when two vertices of the quadrilateral are coincided.

The proof of the second part of the theorem is much simpler. By direct calculations we obtain that

$$Q(a,b) - (2k^2 + 2k + 1)P(a,b)$$
$$= k^4 (a+b)(bk + 1 - a - ak)^2 (b + bk - 1 + ak - 2k)^2 .$$

We note that the equality holds on the straight lines

$$bk + 1 - a - ak = 0, \qquad b + bk - 1 + ak - 2k = 0,$$

the point $(1,1)$ satisfies both these equations and corresponds to the case when the quadrilateral $ABCD$ is a parallelogram. Thus, the above theorem is completely proved.

It should be noted that in the papers [5][6] and [10], [7] the authors studied the corresponding analogue of the Popoviciu's problem for pentagons on the Euclidean plane.

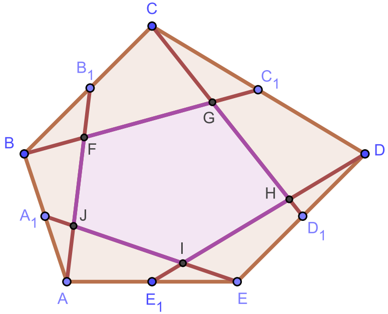

Fig. 3: The Popoviciu's problems for pentagons.

[6] F.A. Dudkin, *Popoviciu's problem for a convex pentagon*, Tr. Rubtsovsk. Ind. Inst., 12 (2003), 31–38 (in Russian), **Zbl.**1036.52002.

[7] K.O. Kizbikenov, *Popoviciu's problem for a convex pentagon*, Vestnik Altaiskogo Gosudarstvennogo Pedagogiceskogo Universiteta, 20 (2014), 16–23 (in Russian).

Let $ABCDE$ be any convex quadrilateral in the Euclidean plane. Let us consider the points A_1, B_1, C_1, D_1, and E_1 of the segments AB, BC, CD, DE, and EA respectively, such that

$$\frac{|AA_1|}{|A_1B|} = \frac{|BB_1|}{|B_1C|} = \frac{|CC_1|}{|C_1D|} = \frac{|DD_1|}{|D_1E|} = \frac{|EE_1|}{|E_1A|} = 1.$$

The straight lines AB_1, BC_1, CD_1, DE_1, EA_1 form a pentagon $FGHIJ$ situated inside $ABCDE$. Let us denote by S and s the areas of the quadrilaterals $ABCDE$ and $FGHIJ$ respectively. The problem consists in the following: to find the best possible constants C_1 and C_2 such that

$$C_1 \cdot S \leq s \leq C_2 \cdot S$$

and to study all cases of equalities, see Fig. 3.

Theorem (Dudkin – Kizbikenov, [5, 10])

For an arbitrary convex pentagon in the Euclidean plane, the inequality

$$\frac{1}{6} \cdot S \leq s < \frac{51}{100} \cdot S$$

holds. Moreover, if $S > 0$ then the equality $6\,s = S$ is fulfilled exactly for pentagons with three coinciding vertices.

Hence, we see that $C_1 = 1/6$. The inequality $\frac{1}{6} \cdot S \leq s$ was proved by F. Dudkin, while he obtained a weaker result for the upper bound: $s < \frac{13}{25} \cdot S$. The inequality $s < \frac{51}{100} \cdot S$ was obtained by K. Kizbikenov. Unfortunately, we do not yet know the exact value of the constant C_2. There are pentagons such that $s > \frac{127}{259} \cdot S$. It is possible that $C_2 = 1/2$.

A similar problem can be considered for a convex n-gon $ABCDEF...$, $n \geq 6$, It should only be noted that the upper estimate becomes trivial, since in the case of degeneration into a triangle $A = B$, $C = D$, and $E = F = \dots$, the equality $s = S$ holds (both polygons degenerate into the same triangle).

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The self Chebyshev radius of the boundary of a convex figure and related extremal problems

For a given metric space (X, d), we denote by $B(x, r)$ the closed ball with center x and radius r. Given a nonempty bounded subset M of X and a nonempty set $Y \subset X$, *the relative Chebyshev radius* (of the set M with respect to Y) is defined by

$$r_Y(M) := \inf_{x \in Y} r(x, M),$$

where

$$r(x, M) := \inf\{r \geq 0 \,|\, M \subset B(x, r)\} = \sup_{y \in M} d(x, y).$$

In the case $Y = M$, we get the definition of *the relative Chebyshev radius of* M *with respect to* M *itself* or *the self Chebyshev radius of the set* M:

$$r_M(M) = \inf_{x \in M} \sup_{y \in M} d(x, y).$$

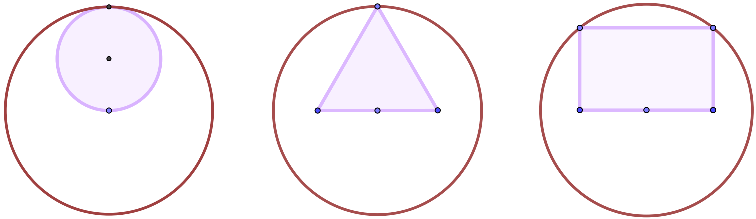

Fig. 4: The self Chebyshev radius for some convex figures and their boundaries.

For brevity we will denote by $\delta(M)$ the self Chebyshev radius $r_M(M)$ of M. It is clear that this value depends only on the set M and the restriction of the metric d to this set, hence it is an intrinsic characteristic of the (bounded) metric space $(M, d|_{M\times M})$. It has also the following obvious geometric sense for compact M: $\delta(M)$ is the smallest radius of a ball having its center in M and covering M.

The study of extremal problems for close convex curves in the Euclidean plane with a given self Chebyshev radius started with the paper [28][8] by Rolf Walter (2017). In particular, he conjectured that $L(\Gamma) \geq \pi \cdot \delta(\Gamma)$ for any closed convex curve Γ in the Euclidean plane, where $L(\Gamma)$ is the length of Γ and d is the standard restricted Euclidean metric. In [28], this conjecture is proved for the case that Γ is a convex curve of class C^2 and all curvature centers of Γ lie in the interior of Γ. It is also shown that the equality $L(\Gamma) = \pi \cdot \delta(\Gamma)$ in this case holds if and only if γ is of constant width.

[8] R. Walter, *On a minimax problem for ovals,* Minimax Theory Appl. 2(2) (2017), 285–318.

As the authors of [2] discovered somewhat later, the original problem from [28], which was to prove the inequality $L(\Gamma) \geq \pi \cdot \delta(\Gamma)$ for any closed convex curve Γ in the Euclidean plane, had already been solved many years ago in paper [6] by K.J. Falconer (the results of that paper were formulated in other terms, without using the self Chebyshev radius). An exposition of the corresponding result can also be found in [14, Theorem 4.3.2]. It should be noted that a stronger result is also proved in [6][9]:

Theorem (Falconer)

Let Γ be a closed rectifiable curve in $\mathbb{R}^n$ (with the Euclidean metric) such that for every point x on Γ there is a point of Γ at distance at least 1 from x. Then Γ has length at least π, this value being attained if and only if Γ bounds a plane convex set of constant width 1.

[9] K.J. Falconer, *A characterisation of plane curves of constant width*, J. Lond. Math. Soc., II. Ser. 16 (1977), 536–538.

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It is also proved in [28] that all C^2-smooth convex curves have good approximations by polygonal chains in the sense of the self Chebyshev radius. This observation leads to natural extremal problems for convex polygons. In particular, the following result given in [28] holds:

Theorem (Walter)

For each triangle P in the Euclidean plane, one has $L(\Gamma) \geq 2\sqrt{3} \cdot \delta(\Gamma)$ with equality exactly for equilateral triangles, where Γ is the boundary of P.

The proof of this result was simplified in the paper [2][10] by V. Balestro, H. Martini, Yu.G. Nikonorov, and Yu.V. Nikonorova, where the authors determined (in particular) the self Chebyshev radius for the boundary of an arbitrary triangle.

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It is also proved in [28] that all C^2-smooth convex curves have good approximations by polygonal chains in the sense of the self Chebyshev radius. This observation leads to natural extremal problems for convex polygons. In particular, the following result given in [28] holds:

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Theorem ([2])

Let P be a triangle in the Euclidean plane with side lengths $a \geq b \geq c$ and with angles $\alpha \geq \beta \geq \gamma$, Γ be the boundary of P. Then the following formula holds:

$$\delta(\Gamma) = \begin{cases} \dfrac{a}{2} & for\, \alpha \geq \pi/2, \\ b\sin(\gamma) & \text{for } \gamma \geq \pi/4, \\ \dfrac{b}{2\cos(\gamma)} & \text{for } \gamma \leq \pi/4 \text{ and } \alpha \leq \pi/2\,. \end{cases}$$

Moreover, some related problems were considered in detail in [2]. In particular, the maximal possible perimeter for convex curves and boundaries of convex n-gons with a given self Chebyshev radius were found.

A half-disk in the Euclidean plane $\mathbb{R}^2$ is a set which is isometric to

$$HD(r) = \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0,\, x^2 + y^2 \leq r^2\}$$

for some fixed $r > 0$. The boundary of $HD(r)$ is the union of a half-circle of radius r and a line segment of length $2r$. Recall that $L(\Gamma)$ means the length of a given convex curve Γ.

Lemma ([2])

Let Γ be the boundary of some half-disk of radius r in $\mathbb{R}^2$. Then $\delta(\Gamma) = r$ and $L(\Gamma) = (2+\pi)\cdot\delta(\Gamma)$.

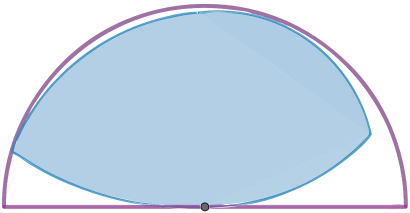

Fig. 5: The estimation from the above for the length of Γ with a given $\delta(\Gamma)$.

Theorem ([2])

For any closed convex curve Γ in the Euclidean plane, one has

$$L(\Gamma) \leq (2+\pi) \cdot \delta(\Gamma),$$

with equality exactly for boundaries of half-disks.

Now we are going to describe all convex n-gons P $(n \geq 2)$ of maximal perimeter among all convex n-gons with the same value of $r = \delta(\Gamma)$, where Γ is the boundary of P. For any convex polygon P we denote by $\mathrm{bd}(P)$ its boundary. At first, we consider an explicit construction of a special family of convex n-gons U_n for $n \geq 2$.

Let us consider a regular $2(n-1)$-gon P_n, inscribed in a circle of radius $r > 0$. Take points $A, B \in P_n$ that are opposite vertices of this polygon (i. e., $d(A,B) = 2r$). Now, consider one of the two half-planes determined by the straight line AB, say H, and consider the union U_n of the line segment $[A,B]$ with the polygonal line $P_n \cap H$. This is an n-gon inscribed in the boundary of the half-disk $\{x \in \mathbb{R}^2 \,|\, d(x,o) \leq r\} \cap H$, where o is the midpoint of the line segment $[A,B]$. It is easy to check that $\delta(U_n) = r$. For $n = 2$ we see that $P_2 = U_2$ is a line segment of length $2r$.

Fig. 6: The polygons U_n for $n = 3$, $n = 4$, and $n = 6$.

Now we are going to describe all convex n-gons P ($n \geq 2$) of maximal perimeter among all convex n-gons with the same value of $r = \delta(\Gamma)$, where Γ is the boundary of P. For any convex polygon P we denote by $\mathrm{bd}(P)$ its boundary. At first, we consider an explicit construction of a special family of convex n-gons U_n for $n \geq 2$.

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Fig. 6: The polygons U_n for $n = 3$, $n = 4$, and $n = 6$.

The perimeter $L(\mathrm{bd}(U_n))$ of U_n is equal to $\lambda_n \cdot r$, where

$$\lambda_n = 2\left(1 + (n-1)\sin\left(\frac{\pi}{2(n-1)}\right)\right)$$

for $n \geq 2$. Note that $\lambda_2 = 4$, since $L(\mathrm{bd}(U_2))$ is the double length of the line segment U_2. It is also easy to see that $\lambda_3 = 2(1+\sqrt{2})$, $\lambda_4 = 5$, and $\lambda_n \to 2 + \pi$ as $n \to \infty$.

Theorem ([2])

For any convex n-gon $P \subset \mathbb{R}^2$ with the boundary Γ, $n \geq 2$, one has

$$L(\Gamma) \leq 2\left(1 + (n-1)\sin\left(\frac{\pi}{2(n-1)}\right)\right) \cdot \delta(\Gamma),$$

with equality exactly for the n-gon U_n defined above.

It is interesting also to find all n-gons with the smallest perimeter among all convex n-gon which boundaries have a given value of the self Chebyshev radius (we call such polygons *extremal*), i. e. $L(\Gamma)/\delta(\Gamma)$ has a minimal possible value, where $\Gamma = \mathrm{bd}(P)$.

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Rolf Walter's conjecture on "magic kites" and related problems

Let us recall that n-gon P is *extremal* if it has the smallest perimeter among all convex n-gon which boundaries have a given value of the self Chebyshev radius (i. e. $L(\Gamma)/\delta(\Gamma)$ has a minimal possible value, where $\Gamma = \operatorname{bd}(P)$).
Extremal triangles are exactly regular triangles, as it was proved by R. Walter in [28][11]. In the same paper [28], Rolf Walter *conjectured* that

$$L(\Gamma) \geq \frac{4}{3}\sqrt{2\sqrt{3}+3} \cdot \delta(\Gamma)$$

for any convex quadrangle $P \subset \mathbb{R}^2$ with the boundary $\operatorname{bd}(P) = \Gamma$. Note that $2+\sqrt{2} > \frac{4}{3}\sqrt{2\sqrt{3}+3} = 3.389946...$.
Note that this inequality becomes an equality for quadrangles P called "magic kites". This definition is taken from [28] and means convex quadrangles which are hypothetically extreme with respect to the self Chebyshev radius.

[11] R. Walter, *On a minimax problem for ovals,* Minimax Theory Appl. 2(2) (2017), 285–318.

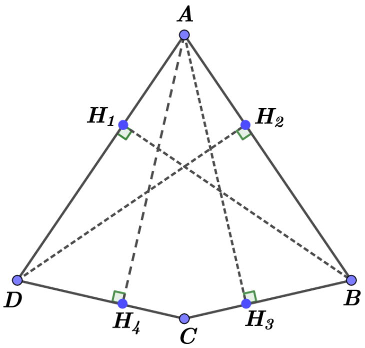

Fig. 7: A magic kite.

Up to similarity, such a quadrangle $ABCD$ could be represented by its vertices, that are as follows (see Fig. 7):

$$A = \left(0, \frac{\sqrt{3}}{3}\sqrt{2\sqrt{3}+3}\right), \quad B = (1,0), \quad C = \left(0, -\frac{1}{3}\sqrt{2\sqrt{3}-3}\right), \quad D = (-1,0).$$

The points H_1, H_2, H_3, H_4 are self Chebyshev centers for the boundary $\Gamma = \mathrm{bd}(ABCD)$ of the quadrangle $ABCD$ and $\delta(\mathrm{bd}(ABCD)) = d(H_2, A) = d(H_1, C) = d(H_3, B) = d(H_4, B)$ is the self Chebyshev radius of its boundary.

The above conjecture by Rolf Walter was confirmed by E.V. Nikitenko and Yu.G. Nikonorov in [16].[12]
Now, we are going to discuss how it is possible to determine all extremal quadrilaterals (4-gons) according to [16].

Let us consider the main steps to prove $L(\Gamma) \geq \frac{4}{3}\sqrt{2\sqrt{3}+3} \cdot \delta(\Gamma)$ for any extremal quadrilateral P. For this goal, we consider some auxiliary problems. We consider a convex quadrangle $P = A_1A_2A_3A_4$ (indices are from $\mathbb{Z}_4$) that is extremal, $\Gamma := \mathrm{bd}(P)$, $\delta(\Gamma)$ is the self Chebyshev radius of Γ. We denote by $G_{\min}$ the set of all point $x \in \Gamma$ such that Γ is a subset of the disc with center x and radius $\delta(\Gamma)$ (i. e. x is a self Chebyshev center of Γ). For any $x \in G_{\min}$ we consider the set $F(x) = \{y \in \Gamma \,|\, d(x, y) = \delta(\Gamma)\}$.

[12] E.V. Nikitenko, Yu.G. Nikonorov, *The extreme polygons for the self Chebyshev radius of the boundary*, Preprint, 2022, arXiv:2301.03218.

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Proposition

Suppose that a quadrilateral P is extremal. If $A \in G_{\min}$ and $A \in [A_i, A_{i+1}]$, then $A_i, A_{i+1} \notin F(A)$. In particular, $F(A) \subset \{A_{i+2}, A_{i+3}\}$.

Let us denote by NE *the quantity of points in the set* $G_{\min}$, or, in other words, the quantity of self Chebyshev centers for Γ. For any $i \in \mathbb{Z}_4$, the set $G_{\min} \cap [A_i, A_{i+1}]$ is either empty or has exactly one point. On the other hand, $G_{\min} \neq \emptyset$ and, therefore, $1 \leq NE \leq 4$. It is easy to prove that $G_{\min}$ does not contain any vertex of P. It is natural to consider all possible values of NE and consider separately every case with the cardinality k, $k = 1, 2, 3, 4$, for the set $G_{\min}$. We will denote every such case as **Case** k.

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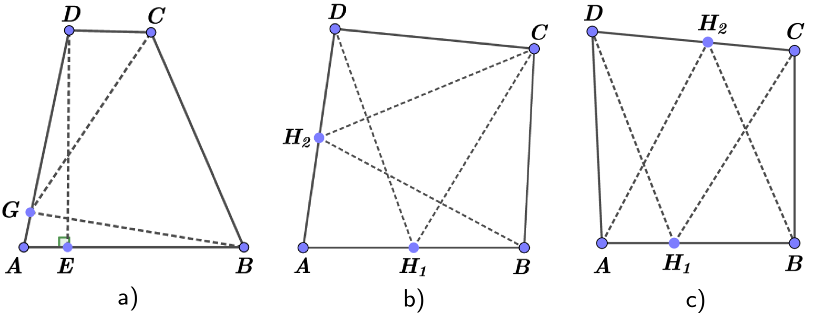

Fig. 8: a) Case 2.1, b) Case 2.2, c) Case 2.3.

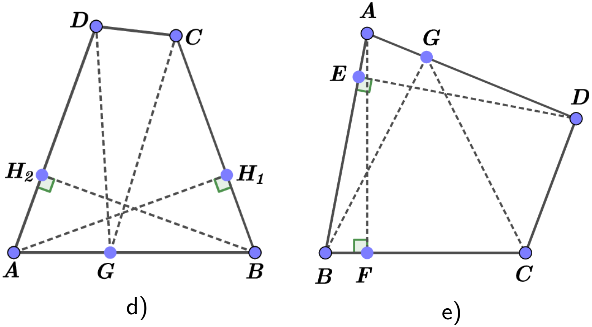

Fig. 9: d) Case 3.1, e) Case 3.2.

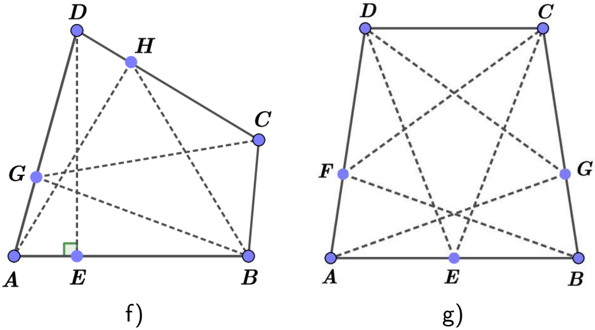

Fig. 10: f) Case 3.3, g) Case 3.4.

In [16] (this paper has 40 pages), all cases were considered in detail. In Fig. 8, Fig. 9, and Fig. 10, all the essential subcases of Cases 2 and 3 are shown.

In the final part of [16], the authors considered some results of numerical calculations. The search for possible extremal polygons for small values of n was undertaken by E.V. Nikitenko. His experiments led to the conjecture that *for odd n, any extremal polygon is a regular n-gon* (this is the case for $n = 3$). On the other hand, *for even values of n, regular n-gons are not extremal* in the above sense.

The type of an extremal polygon (quite possibly) depends on the power of the number 2 with which it enters to n as a multiplier. See Fig. 11 for hypothetical extreme polygons for $n = 6$ and $n = 10$. The calculation of the characteristics of these polygons (in particular, these polygons are equilateral), as well as numerous computer experiments, lead to the following conjectures:

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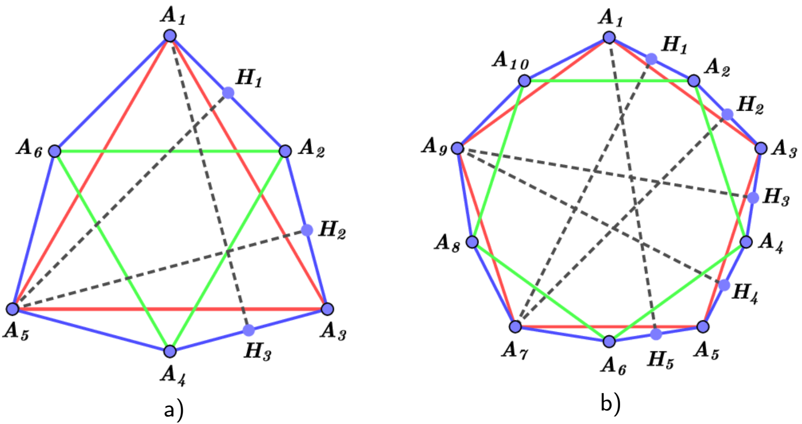

Fig. 11: Extreme n-gon candidates for: a) $n = 6$; b) $n = 10$. In both cases, the green and red polygons are regular, and are negative homothets of each other.

Conjecture

For any convex 6-gon P in the Euclidean plane, one has

$$L(\Gamma) \geq 12\left(2 - \sqrt{3}\right) \cdot \delta(\Gamma),$$

where Γ is the boundary of P.

Conjecture

For any convex 10-gon P in the Euclidean plane, one has

$$L(\Gamma) \geq 20\left(1 + \sqrt{5} - \sqrt{5 + 2\sqrt{5}}\right) \cdot \delta(\Gamma),$$

where Γ is the boundary of P.

Now, we consider several extremal problems on the Euclidean plane, the interest in which is due, among other things, to the study of the properties of the self Chebyshev radii of the boundaries of convex polygons.

A side AB of a convex polygon P is said to have an l-strut if there exists a point $C \in P$ such that such that $d(A, C) = d(B, C) = l$. It should be noted that C should not be a vertex of P in this definition. For $l = 1$ we will use the term *strut* instead of 1-strut.

A convex polygon P in Euclidean plane is said to have *the $\Delta(l)$ property* if every its side has an l-strut (i. e. for the endpoints A and B of every side of P, there is a point $C \in P$ such that $d(A, C) = d(B, C) = l$). Let us recall Problem 1 from [2].[13]

Problem ([2])

Given a real number $l > 0$ and a natural number $n \geq 3$, determine the best possible constant $C(n, l)$ such that the inequality $L(P) \geq C(n, l)$ holds for the perimeter $L(P)$ of every convex polygon P with n vertices that satisfies the $\Delta(l)$ property.

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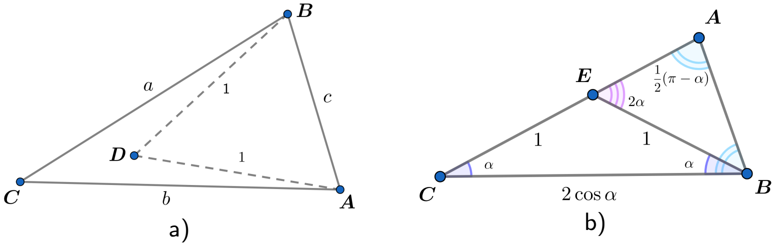

Fig. 12: a) A triangle with the Δ property; b) A "narrow" isosceles triangle with the Δ property.

It is easy to show that $C(3, l) = 3l$. The solution of the above problem for $n \geq 4$ was obtained recently in the paper [18][14] by Yu.G. Nikonorov and O.Yu. Nikonorova. It is clear, using similarities in the Euclidean plane, we may restrict our attention to the case $l = 1$. In what follows, we consider only the case $l = 1$ and, for brevity, we will call the $\Delta(1)$ property *the Δ property*.

[14] Yu.G. Nikonorov, O.Yu. Nikonorova, *Some extremal problems for polygons in the Euclidean plane*, Preprint, 2022, arXiv:2209.05940.

One of the main results of [18] is as follows.

Theorem ([18])

Given a natural number $n \geq 3$, the perimeter $L(P)$ of any polygon P with n vertices, that satisfies the Δ property, is such that $L(P) \geq 3$. Moreover, this inequality cannot be improved, while the equality $L(P) = 3$ holds if and only if P is a regular triangle with unit side.

A regular triangle with unit side satisfies the Δ property and has perimeter 3 (the smallest possible). It should be noted that a triangle ABC with the Δ property should not contain a regular triangle with unit side. To show this let us consider a "narrow" isosceles triangle ABC such that $\angle ACB = \alpha$, $|BA| = |CA| = 2\cos\alpha$. It is easy to see that this triangle satisfies the Δ property if $\alpha \leq \pi/3$, see Fig. 12 b). On the other hand, such triangle does not contain a regular triangle with unit side for sufficiently small $\alpha > 0$.

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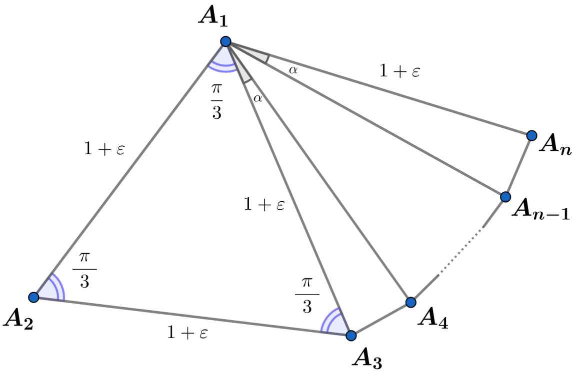

Fig. 13: An n-gon with the Δ property for $n \geq 4$.

Let us show that for any given $n \geq 4$, there are n-gons P with the Δ property such that the perimeter $L(P)$ is arbitrarily close to 3. We construct such polygons in a small neighborhood (with respect to the Hausdorff metric) of a regular triangle with unit side. Let us fix a small number $\varepsilon > 0$. We start from the regular triangle $A_1A_2A_3$ with side length $1+\varepsilon$.

Further, we fix a number $\alpha > 0$ such that $\alpha < \min\{\pi/n, \varepsilon\}$ and consider the points $A_4, A_5, \dots, A_{n-1}, A_n$ in the plane such that $|A_1A_i| = 1+\varepsilon$ and $\angle A_{i-1}A_1A_i = \alpha$, $i = 4, 5 \dots, n$. Let P be the convex hull of the points A_i, $i = 1, 2, \dots, n$, see Fig. 13. For any side A_iA_{i+1} of P with $i = 1, 2, \dots, n-1$ we can easily choose a point $B \in P$ such that $|A_iB| = |A_{i+1}B| = 1$. The same is true for the side A_1A_n for sufficiently small $\alpha > 0$, i. e., the polygon P has the Δ property for sufficiently small $\alpha > 0$. For the perimeter of P we have the following estimate:

$$\begin{aligned} L(P) &= |A_1A_n| + |A_1A_2| + |A_2A_3| + \sum_{i=3}^{n-1} |A_iA_{i+1}| = 3 + (n-3)|A_3A_4| \\ &= (1+\varepsilon)\cdot\big(3 + 2(n-3)\sin(\alpha/2)\big) < (1+\varepsilon)\cdot\big(3+(n-3)\alpha\big). \end{aligned}$$

Now, it is clear that for sufficiently small $\varepsilon > 0$ and $\alpha > 0$, $L(P)$ is as close to 3 as we want.

If X and Y are sets in $\mathbb{R}^2$, then $X+Y=\{x+y\,|\,x\in X, y\in Y\}$ denotes *the Minkowski sum* of X and Y. For given $t\in\mathbb{R}$ and $X\subset\mathbb{R}^2$, we consider $tX:=\{t\cdot x\,|\,x\in X\}$. In what follows, we will often use $-X=\{-x\,|\,x\in X\}$.

We will need a special transformation of convex polygons in $\mathbb{R}^2$. For a convex polygon P, we define *the difference body* by the formula

$$D(P)=P+(-P).$$

Note that *the central symmetral* of P, defined by

$$\Diamond(P)=\frac{1}{2}P+\frac{1}{2}(-P),$$

differs from the difference body $D(P)$ only by a dilatation factor of $1/2$, see e. g. [3]. It should be noted that the central symmetral have many important properties. For instance, it is easy to check that the perimeters of the polygons P and $\Diamond(P)$ coincide (hence, the perimeter of $D(P)$ is in two times greater).

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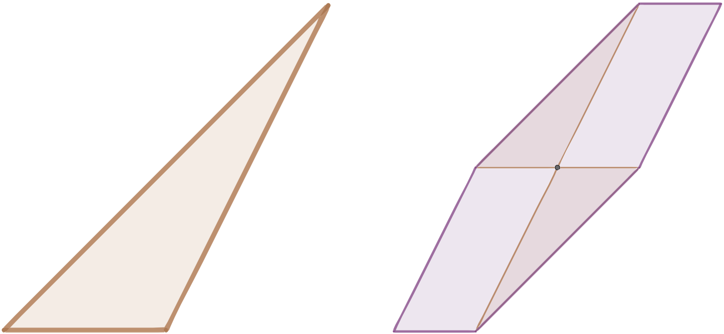

Fig. 14: A triangle P and the central symmetral $\Diamond(P)$ of P

It is clear that the difference body $D(P) = P + (-P)$ of any convex polygon P is a centrally symmetric polygon. In particular, the origin O is the center of $D(P)$. It is remarkable that the difference body of a polygon P with the Δ property also has one special property.

We will say that an origin-symmetric (i. e. centrally symmetric with the origin O as the center) polygon P has *the* Δ^s *property* if for any its side AB, P contains a rectangle $KLMN$ such that $\overrightarrow{KL} = \overrightarrow{NM} = \overrightarrow{AB}$ and the distance from O to any vertex of $KLMN$ is 1. The following results are very important.

Proposition

Let P be a polygon with the Δ property. Then its difference body $D(P) = P + (-P)$ has the Δ^s property. Moreover, the perimeter $L(D(P))$ is equal to $2 \cdot L(P)$.

Proposition

Let P be an origin-symmetric convex polygon with the Δ^s property and such that any side of it is not longer than 1. Then the perimeter $L(P)$ is not less than 6. Moreover, $L(P) = 6$ if and only if P is a regular hexagon with side of length 1.

In fact, by passing to a difference body, the proof of the desired result was reduced to verifying a much simpler statement for an origin-symmetric polygon.

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It is reasonable to try to weaken the conditions of the previous theorem. The most natural variant is to prescribe the presence of struts not on all sides of the polygon, but *only on some of them*.

The presence of a strut for only one side of the polygon P, is not enough for the inequality $L(P) \geq 3$. Indeed, we can consider triangles ABC with $|AC| = |BC| = 1$ and with very short AB. In this case the side AB has a strut, but $L(P)$ can be smaller than $2 + \varepsilon$ for any given $\epsilon > 0$. On the other hand, if we assume that a side AB has a strut and $|AB| \geq 1$, then we get $L(P) \geq 3$ obviously.

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It is interesting that a similar idea works in the case of two adjacent to each other sides of P, but the corresponding result is much more difficult to obtain. The second main result of [18] is as follows.

Theorem ([18])

Given a natural number $n \geq 3$, let P be a convex polygon with the consecutive vertices $A_1, A_2, A_3, \dots, A_{n-1}, A_n$ such that the sides A_1A_2 and A_2A_3 have struts and $|A_1A_2| + |A_2A_3| \geq 1$. Then the perimeter $L(P)$ of P satisfies the inequality $L(P) \geq 3$.

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The proof of this theorem is based on the using of polynomial ideals, the ability to eliminate some variables and to find the corresponding Gröbner bases, see, e. g., [4]. It is reasonable to perform all necessary calculations using some standard system of symbolic calculations.

It should be noted that a regular triangle with unit side is not a unique polygon with $L(P)=3$ in the above theorem. There are a continuous family of quadrangle with this property and one distinguished pentagon, see Fig. 15.

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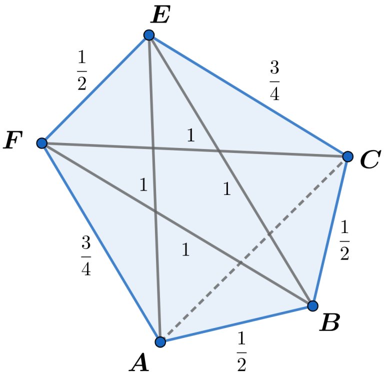

Fig. 15: A special pentagon P with perimeter 3.

This pentagon is quite remarkable. Note that $\angle CEB = \angle CFB = \angle CAB = \angle AFB = \angle AEB = \angle ACB = \angle ECF = \angle EBF = \angle EAF = 2\arcsin(1/4) = \arccos(7/8)$. In particular, all vertices of this pentagon lie on the same circle. Moreover, $|AC| = 7/8$. Thus, if we increase this pentagon by 8 times, we get an *integer pentagon*, in which all sides and diagonals have integer lengths. This pentagon was found at first in [15], see also the discussion in [21, P. 19].

The above theorems and examples of polygons with the perimeters close to 3 naturally entail the following

Conjecture ([18])

Given natural numbers $n \geq 3$ and $m = 1, \dots, n-1$, let P be a convex polygon P with the consecutive vertices $A_1, A_2, A_3, \dots, A_{n-1}, A_n$ such that the sides $A_1A_2, A_2A_3, \dots, A_mA_{m+1}$ have struts and $|A_1A_2| + |A_2A_3| + \dots + |A_mA_{m+1}| \geq 1$. Then the perimeter $L(P)$ of P satisfies the inequality $L(P) \geq 3$.

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The self-perimeter of the unit ball of the Minkowski plane and László Fejes Tóth's conjecture

The first topic is related to the *self-perimeter of the unit ball of the Minkowski plane*.
We discuss well known results by S. Golab (S. Gołąb) [8],[15] Yu.G. Reshetnyak [23],[16] D. Laugwitz [11],[17] and J.J. Schäffer [24][18] in this direction, see also [27].

Let A^2 be an affine plane and B a convex figure on A^2 (i. e. a compact convex subset of A^2 with non-trivial interior) containing a point O in its interior (not necessarily symmetric about O). Without loss of generality, we may identify A^2 with $\mathbb{R}^2$ and assume that O is the origin in $\mathbb{R}^2$. Let us suppose also that $\mathbb{R}^2$ is supplied with a scalar product with the corresponding norm $|\cdot|$.

---

[15] S. Golab, *Quelques problèmes métriques de la géométrie de Minkowski*, Trav. Acad. Mines Cracovie Fasc. 6 (1932) 1–79 (in Polish).

[16] Yu.G. Reshetnyak, *An extremum problem from the theory of convex curves*, Uspekhi Mat. Nauk [Russian Math. Surveys] 8(6) (1953), 125–126 (in Russian).

[17] D. Laugwitz, *Konvexe Mittelpunktsbereiche und normierte Räume*, Math. Z. 61 (1954), 235–244 (in German).

[18] J.J. Schäffer, *Inner diameter, perimeter, and girth of spheres*, Math. Ann. 173 (1967), 59–79, addendum ibid. 173 (1967), 79–82.

For distinct points $x, y \in \mathbb{R}^2$, we define the distance $d_B(x, y)$ from x to y as follows. Let us consider a ray from O in the direction of the vector $y - x$ and let z be the point of its intersection with the boundary of B. Now, we put

$$d_B(x, y) = |y - x|/|z|.$$

Thus, d_B is a (possibly, nonsymmetric) metric. Obviously, the distance $d_B(x, y)$ does not depend on the choice of a scalar product on $\mathbb{R}^2$.
An affine plane A^2 with the metric d_B (possibly, nonsymmetric) introduced above is called a *Minkowski plane* M^2, whereas the figure B is called the *norming figure* or the *unit disk* of M^2. If O is the center of B, then we have a classical Minkowski plane M^2 with the symmetric norm $\|x\|_B = d_B(O, x)$.

Let P be a convex bounded polygon on M^2. Denote by $L_B^+(P)$ and by $L_B^-(P)$ its perimeters traversed counterclockwise and clockwise, respectively. For a compact convex figure K, we define the *perimeters* by the formula

$$U_B^{\pm}(K) = \sup L_B^{\pm}(P),$$

where the supremum is taken over all convex polygons P within K. It is not so difficult to check that the perimeters $U^{\pm}$ have the monotonicity property: If $K_1 \subset K_2$, then $U_B^{\pm}(K_1) \leq U_B^{\pm}(K_2)$, and the equality holds if and only if $K_1 = K_2$.

If O is not the center of B, then there is a convex figure K such that $U^+(K) \neq U^-(K)$. If we consider $B' := \{(x,y) \in \mathbb{R}^2 \,|\, (-x,y) \in B\}$, then $U_{B'}^+(K) = U_B^-(K)$ and $U_{B'}^-(K) = U_B^+(K)$ for any convex figure K. On the Minkowski plane with a norming figure B symmetric about O, both perimeters have a common value $U(K)$ for any convex figure K.

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It is reasonable and interesting to consider and study the perimeters $U_B^{\pm}$ for the unit disk B. In this case we deal with the *self-perimeters* $U_B^{\pm}(B)$ for B. If O is the center of B, then we have (in fact) only one *self-perimeter* $U_B(B) = U_B^{+}(B) = U_B^{-}(B)$.

We have the following classical result.

Theorem (Golab – Reshetnyak – Laugwitz – Schäffer)

For any Minkowski plane with a centrally symmetric unit disk B, the self-perimeter $U_B(B)$ of B satisfy the inequality $6 \leq U_B(B) \leq 8$. Moreover, $U_B(B) = 6$ if and only if B is an affinely regular hexagon and $U_B(B) = 8$ if and only if B is a parallelogram.

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A little about the history of this theorem. S. Golab proved in 1932 that the *self-perimeter* $U(B)$ satisfies $6 \leq U_B(B) \leq 8$. Yu.G. Reshetnyak [23] in 1953 and D. Laugwitz [11] in 1954 rediscovered Golab's result. In 1967, J.J. Schäffer [24] proved that $U(B) = 6$ only for an affinely regular hexagon, and $U(B) = 8$ only for a parallelogram.

For any Euclidean norm, the self-perimeter is $2\pi = 6{,}283185...$ obviously. Let us consider some hints on how to prove the inequality $6 \leq U_B(B) \leq 8$.

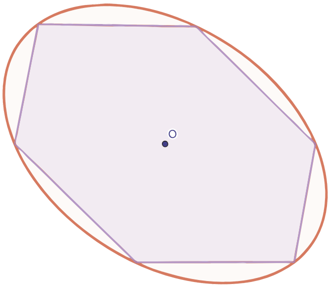
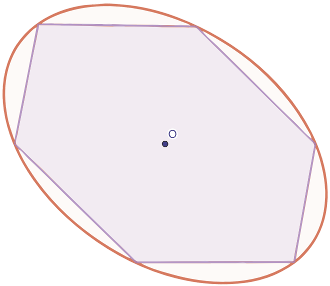


Fig. 16: The key idea of the proof of the inequality $U_B(B) \geq 6$.

A hint for proving the inequality $U_B(B) \geq 6$. One can consider an affinely regular hexagon HP inscribed in B, that is, an affine image of a regular hexagon with vertices on the boundaries of B. It is easy to construct such a hexagon. It is clear that O is its center and the perimeter $U_B(HP)$ of this hexagon is equal to 6 (each side of HP has the unit length). By the monotonicity property of the perimeters, we obtain $U_B(B) \geq U_B(HP) = 6$.



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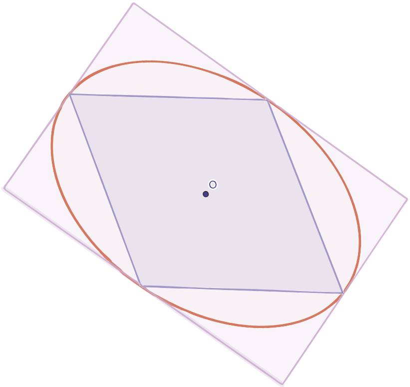

Fig. 17: The key idea of the proof of the inequality $U_B(B) \leq 8$.

A hint for proving the inequality $U_B(B) \leq 8$. One can consider an parallelogram IP inscribed in B (all vertices of IP are on the boundaries of B) with maximal area among all such parallelograms. It is clear that the straight lines through each vertex of IP parallel to the diagonal of IP, which is not incident to a given vertex, is a support line for B. Four such straight lines generate a parallelogram OP which perimeter $U_B(OP)$ is 8 (each side of OP has length 2). By the monotonicity property of the perimeters, we obtain $U_B(B) \leq U_B(OP) = 8$.

It should be noted that the original proofs of the Golab – Reshetnyak – Laugwitz and Schäffer results were essentially based on the symmetry of B with respect to the origin O.

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Let us consider the case of not necessarily symmetric norming figure B (the point O is in the interior of B). We have the following result.

Theorem (Shcherba)

For any Minkowski plane, the self-perimeters $U_B^{\pm}(B)$ of the unit disk B satisfy the inequality $U_B^{\pm}(B) \geq 6$. Moreover, $U^-(B) = 6$ (or $U^+(B) = 6$) if and only if B is an affinely regular hexagon. In particular, $U_B^+(B) = 6$ is equivalent to $U_B^-(B) = 6$.

A.I. Shcherba proved the inequality $U^{\pm}(B) \geq 6$ in 2003 [25][19] (the corresponding problem was stated by Golab in 1932 [8]). Moreover, in 2007, A.I. Shcherba proved that the equality $U^-(B) = 6$ or $U^+(B) = 6$ holds if and only if B is an affinely regular hexagon, see [26][20].

[19] A.I. Shcherba, *On an estimation of the perimeter of the unit circle on the Minkowski plane*, Tr. Rubtsovskogo Industrial. Inst., 12 (2003), 96–107 (in Russian).
[20] A.I. Shcherba, *Unit disk of smallest self-perimeter in the Minkowski plane*, Mat. Zametki 81(1) (2007), 125–135 (in Russian).

Some simplest examples show that there is no absolute constant bounding the self-perimeters $U^{\pm}(B)$ from above in the Minkowski plane M^2.

Example

Let us consider the norming figure B that is the quadrangle $KLMN$ on $\mathbb{R}^2$, where $K = (r, 1)$, $L = (r, -1)$, $M = (-1, -1)$, and $N = (-1, 1)$ for some $r > 0$. It is easy to check that

$$U_B^{\pm}(B) = 6 + r + 1/r$$

and

$$U_B^{\pm}(B) \to +\infty \qquad \text{as} \qquad r \to +\infty.$$

On the other hand we have the following result.

Theorem (Grünbaum – Makeev)

For any convex figure B, one can choose a point O in the interior of B so that the self-perimeters satisfy the inequality $U_B^{\pm}(B) \leq 9$. Moreover, this estimate cannot be improved for a triangle.

This theorem was obtained by B. Grünbaum in 1964 [9].[21] In 2003, V.V. Makeev rediscovered this result [12].[22]

[21] B. Grünbaum, *Self-circumference of convex sets*, Colloq. Math. 13 (1964), 55–57.
[22] V.V. Makeev, *On the upper bound of the perimeter of a nonsymmetric unit disk on the Minkowski plane*, Zap. Nauch. Semin. POMI 299 (2003), 262–266 (in Russian).

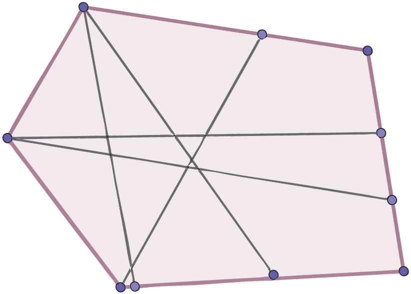

Fig. 18: The longest chords of a polygon parallel to its sides.

Now, we consider one conjecture by László Fejes Tóth.
Let P be a plane convex n-gon with side lengths $a_1, \dots, a_n$. Let b_i be the length of the longest chord of P parallel to the i-th side.

It was conjectured by L. Fejes Tóth in [7][23] that

$$3 \leq \sum_{i=1}^{n} a_i/b_i \leq 4,$$

whereas $\sum_{i=1}^{n} a_i/b_i = 3$ if and only if P is a snub triangle obtained by cutting off three congruent triangles from the corners of a triangle, while $\sum_{i=1}^{n} a_i/b_i = 4$ if and only if P is a parallelogram. This conjecture was partially proven in [7].

A complete solution to this problem was obtained by Yu.G. Nikonorov and N.V. Rasskazova in [20].[24] To show the main idea of this solution we need to introduce some definitions.

[23] L. Fejes Tóth, *Über eine affininvariante Maßzahl bei Eipolyedern*, Stud. Sci. Math. Hung. 5 (1970) 173–180 (in German).

[24] Yu.G. Nikonorov, N.V. Rasskazova, *On a problem of L. Fejes Tóth*, Mat. Tr. 5(1) (2002), 102–113 (in Russian).

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Let us recall the notion of "the central symmetral" for polygons. For a convex polygon P in $\mathbb{R}^2$, *the central symmetral* of P, defined by

$$\Diamond(P) = \frac{1}{2}P + \frac{1}{2}(-P),$$

see e. g. [3]. Recall that the central symmetral have many important properties. For instance, it is easy to check that the perimeters of the polygons P and $\Diamond(P)$ coincide.

Now, we are ready to explain the main idea of solving the above problem by L. Fejes Tóth. It should be noted that the central symmetral was an important element of this solution.

In fact, it was observed in [20], that the inequality $3 \leq \sum_{i=1}^{n} a_i/b_i \leq 4$ follows from the Golab – Reshetnyak – Laugwitz result considered for the central symmetral $\Diamond(P) = \frac{1}{2}P + \frac{1}{2}(-P)$ of P. The corresponding Schäffer result gives all polygons with $\sum_{i=1}^{n} a_i/b_i = 3$ and $\sum_{i=1}^{n} a_i/b_i = 4$. It should be noted also that [20] contains self-contained and modified proofs of the Golab – Reshetnyak – Laugwitz and Schäffer results.

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Thank you for your time and attention!

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